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Future Possible Age of the Universe with Density Variation

Oishi Khanam, Md. Nirab Hossain and Md. Showkat Ali
Department of Applied Mathematics, University of Dhaka, Dhaka 1000, Bangladesh
Email: oishi-2015917302@amath.du.ac.bd

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Abstract: A fundamental principle and assumption of cosmology says that the universe is homogeneous and isotropic when viewed on a large scale. According to the cosmological principle, space might be flat, or have a negative or positive curvature in cosmological model. Positively curved universe denotes the closed universe and negatively curved universe denotes the open universe. Our universe type is flat because it expands in every direction neither curving positively nor negatively. We have observed that the progression of the universe is based on radiation and matter domination. In this paper we also have observed that future possible upper limit age of the universe is 9.4203×10^{10} years which varies with density.

Index Terms: Friedmann equation, Radiation domination, Matter domination, Raychaudhuri equation

1. Introduction

Progression of the universe is a part of general relativity and cosmology. We can see the expansion by cosmological model named as Friedmann model. Friedmann equation is

$$\dot{R}^2 + K = \frac{8}{3}\pi G R^2 \rho \tag{1}$$

Since our universe's expansion type is flat, the curvature $K = 0$. Earlier, the universe was radiation dominated having energy $T = 0$. Later, the universe undergoes matter dominated relating with pressure $p = 0$.

We see that the expansion rate changes depending on the domination factors such as radiation and matter. We find the current age of our universe by Hubble's law. The major research objective is finding the maximum possible upper limit age by using Friedmann and Raychaudhuri equation. We formulate an equation that shows the relation between the age and the density of universe. The formula and the solutions we have observed are more appropriate compared to the existing solutions. Yet there are a lot of limitations. The constant risen in deriving the relation between scale factor and density has been taken as *unity* which is an assumption. Also, we cannot find the density of the universe because of lack of instruments.

2. Literature Review

The work on general relativity started when Einstein stated his theory in 1915. After that a consistent theory of an expanding universe had been established by using Einstein's field equations. Applying the most general principles to the nature of the universe yielded a dynamic solution that conflicted with the then-prevalent notion of a static universe. In 1922, Friedmann equations were derived by Alexander Friedmann from Einstein's field equations, showing that the universe might expand at a rate calculable by the equations. The Friedmann's parameter is the scale factor which is considered as a scale invariant form of the proportionality constant of Hubble's law. Georges Lemaître independently found a similar solution in his 1927 paper discussed in the following section. In succession, Amal Kumar Raychaudhuri established his equations in 1955 by to describe gravitational focusing properties and space-time singularities. P.S. Joshi worked on the upper limit age of the universe in 1986. With accurate values of parameters of the modern approach, we have tried to obtain the future possible age.

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3. Methodology

We have observed the relation among density and scale factor with time from Friedmann equation. Since we know that our universe exists for the attraction of particles, most probably the universe will destroy if the attraction is negligible. Then the density of particles will harshly decline. By this concept, we have calculated the upper limit age which varies with density. The concept of energy momentum tensor has been applied in Einstein equation. We used Raychaudhuri equation throughout this paper to find the age of the universe. Some methods have been generated that require programming. All the results are verified by MATLAB.

4. Scale Factor and Density from Friedmann Equation

In the early universe after the Big Bang the matter of the universe was negligible and it was mostly dependent on the radiation. We know from energy tensor that[7]

From this, we can write $T = 0$, because the universe doesn't depend on matter. Then we have[1]

$$p = \frac{\rho}{3} \tag{2}$$

$$\begin{gathered} T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu} \\ \Rightarrow\ T_{\mu\nu}g^{\mu\nu} = (\rho + p)u_\mu u_\nu g^{\mu\nu} - p g_{\mu\nu}g^{\mu\nu} \\ \Rightarrow\ T = (\rho + p)u_\mu u_\nu - 4p \\ \left[\because T_{\mu\nu}g^{\mu\nu} = T, R_{\mu\nu}g^{\mu\nu} = R, g_{\mu\nu}g^{\mu\nu} = 4\right] \\ \Rightarrow T = (\rho + p) - 4p = \rho - 3p \end{gathered}$$

We know

$$\Rightarrow \dot{\rho} + (\rho + p)\frac{3\dot{R}}{R} = 0 \tag{3}$$

By substituting $p = \frac{\rho}{3}$ in (3), we have

$$\begin{gathered} \dot{\rho} + 4\rho \cdot \frac{\dot{R}}{R} = 0 \\ \Rightarrow\ R^4\dot{\rho} + 4\rho R^3\dot{R} = 0 \\ \Rightarrow\ \frac{d}{dt}(R^4\rho) = 0 \\ \Rightarrow\ R^4\rho = R_0^4\rho_0 = c \end{gathered}$$

$$\rho = \frac{R_0^4\rho_0}{R^4} = \frac{c}{R^4} \tag{4}$$

Dealing with matter dominated universe, the pressure p can be set 0 (as an approximation) to get standard Friedmann models. When compared with density, putting $p = 0$ in (3)

$$\begin{gathered} \dot{\rho} + \rho\frac{3\dot{R}}{R} = 0 \\ \Rightarrow\ \frac{\dot{\rho}}{\rho} = \frac{-3\dot{R}}{R} \\ \Rightarrow\ log\rho = -3logR + c \\ \Rightarrow\ log\rho + logR^3 = c \\ \Rightarrow\ log(\rho R^3) = c \end{gathered}$$

$$\rho R^3 = constant = c \tag{5}$$

Now if we want to see the relation between density and scale factor both for matter and radiation domination, we can draw a graph by considering $c = 1$

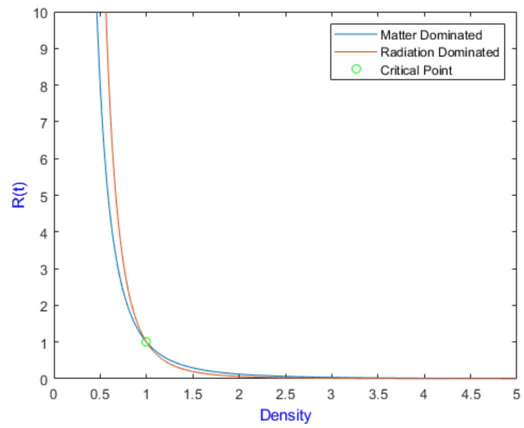

Fig. 1. Scale factor with respect to ratio of density

In the figure, two different curved lines represent different scale factors. Red line indicates the change of scale factor $R(t)$ as density changes when the universe was dominated by radiation. Blue line indicates the change of scale factor $R(t)$ as density changes when the universe is dominated by matter. Green circle denotes the critical point, when these two lines collide. Since the universe is flat, so curvature is zero. From Friedmann equation we know

$$\dot{R}^2 + K = \frac{8}{3}\pi G R^2 \rho \tag{6}$$

Assigning $K = 0$, in the Friedmann equation gives

$$\dot{R}^2 = \frac{8}{3}\pi G \rho R^2$$
$$\Rightarrow \dot{R}^2 = \frac{8}{3}\pi G \frac{R_0^4 \rho_0}{R^4} R^2$$
$$\Rightarrow \dot{R} = \sqrt{\frac{8}{3}\pi G \frac{R_0^4 \rho_0}{R^4} R^2}$$
$$\Rightarrow R\dot{R} = \sqrt{\frac{8}{3}\pi G R_0^4 \rho_0}$$
$$\Rightarrow R\frac{dR}{dt} = \sqrt{\frac{8}{3}\pi G R_0^4 \rho_0}$$
$$\Rightarrow RdR = \sqrt{\frac{8}{3}\pi G R_0^4 \rho_0}dt$$
$$\Rightarrow \frac{R^2}{2} = \sqrt{\frac{8}{3}\pi G R_0^4 \rho_0}t + C$$

We can use the initial condition, at $t = 0$, $R = 0$, $\rho_0 = \rho_{cr} = \frac{3H_0^2}{8\pi G} \Rightarrow C = 0$

$$R = \left(\frac{32}{3}\pi G R_0^4 \cdot \frac{3H_0^2}{8\pi G}\right)^{\frac{1}{4}} \cdot t^{\frac{1}{2}}$$
$$\Rightarrow R = \sqrt{2R_0 H_0} \cdot t^{\frac{1}{2}}$$

$$\therefore R \propto t^{\frac{1}{2}} \tag{7}$$

So, we can say that the scale factor of radiation dominated universe is proportional to $t^{\frac{1}{2}}$. If we consider matter dominated universe, let t_0 be age of the universe today and $R_0 = R(t_0)$, $\rho_0 = \rho(t_0)$ are the today's value of R and ρ. Using (5), we have $\rho R^3 = \rho_0 {R_0}^3$ Now, (6) implies with flat universe assumption $K = 0$

$$\dot{R}^2 = \frac{8\pi G R_0^3 \rho_0}{3R}$$
$$\Rightarrow \dot{R} = \sqrt{\frac{8\pi G R_0^3 \rho_0}{3R}}$$
$$\Rightarrow \sqrt{R} dR = \sqrt{\frac{8}{3}\pi G R_0^3 \rho_0} dt$$
$$\Rightarrow \int \sqrt{R} dR = \int \sqrt{\frac{8}{3}\pi G R_0^3 \rho_0} dt$$
$$\Rightarrow \frac{R^{\frac{3}{2}}}{\frac{3}{2}} = \sqrt{\frac{8}{3}\pi G R_0^3 \rho_0} t + C_1$$

We can use the initial condition, at $t = 0$, $R = 0$, $R_0 = 1$, $\rho_0 = \rho_{cr} = \frac{3H_0^2}{8\pi G}$, so $C_1 = 0$.
Then we have,

$$R(t) = \left(\frac{3}{2}\sqrt{\frac{8}{3}\pi G R_0^3 \rho_0}\right)^{\frac{1}{3}} t^{\frac{2}{3}} + C_1$$
$$\Rightarrow R(t) = \left(\frac{3}{2}\sqrt{\frac{8}{3}\pi G R_0^3 \frac{3H_0^2}{8\pi G}}\right)^{\frac{1}{3}} t^{\frac{2}{3}} + C_1$$
$$\Rightarrow R(t) = \left(\frac{3H_0}{2}\right)^{\frac{2}{3}} t^{\frac{2}{3}}$$
$$\therefore R(t) \propto t^{\frac{2}{3}} \tag{8}$$

So, we can say that the scale factor of matter dominated universe is proportional to $t^{\frac{2}{3}}$.

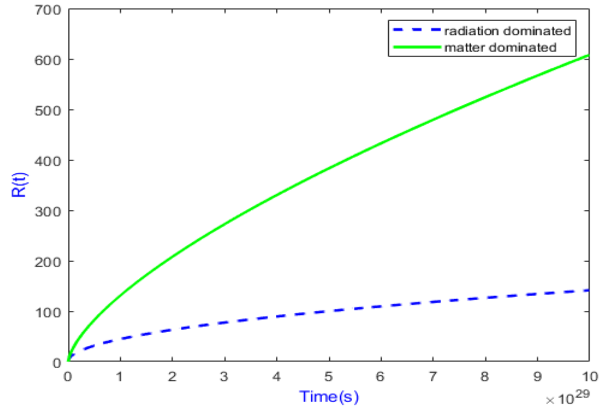

Fig.2. Expansion of our universe

The graph shows that when the universe is radiation dominated, the scale factor is proportional to $t^{\frac{1}{2}}$. Later, when the universe is matter dominated, the scale factor is proportional to $t^{\frac{2}{3}}$ with respect to time. In the graph, the dashed blue line indicates the radiation domination and the solid green line indicates the matter domination.

5. Future Possible Ages of the Universe

If we seek to find the current age of universe, we can calculate it from Hubble's law, $v = H_0 D$, where $D = vt$. We can calculate the age of the universe by now $t = \frac{D}{v} \Rightarrow t = \frac{D}{H_0 D} \Rightarrow t = \frac{1}{H_0}$ $H_0 = 73kms^{-1}/Mpc$ is the best estimation value of Hubble constant. To convert this to an age, we'll need to convert units. Since, $1Mpc$(megaparsec) is equal to $3.08 \times 10^{19} km$

$$H_0 = 73kms^{-1}Mpc \times \frac{1Mpc}{3.08\times10^{19}km} = 2.37 \times 10^{-18} s^{-1}$$

As a result, the Universe's age is[15] $t = \frac{1}{H_0} = \frac{1}{2.37\times10^{-18}s^{-1}} = 4.22 \times 10^{17} s = 13.4$ billion years For finding the upper limit age of the universe, we consider the Raychaudhuri equation[6] as follows:

$$\frac{d\theta}{dt} = -R_{\mu\nu}u^{\mu}u^{\nu} - \frac{1}{n}\theta^2 - 2\sigma^2 + 2\omega^2 \tag{9}$$

Here, $n = 2 \Rightarrow$ null geodesic. $n = 3 \Rightarrow$ time like geodesics. Where u^{μ} is the congruence's time-like tangent vector on the manifold. Raychaudhuri equations use congruent geodesic curves to describe the volume expansion rate. Here, $\theta > 0$ is expnsion, where $\theta = X_i^i$ and X_{ij} is the 2^{nd} fundamental from of the spacelike hyper surface, $\sigma > 0$ is shear and ω is rotation/vorticity tensors. For $\omega = 0$ and $n = 3$ for time like geodesics, (9) becomes

$$\frac{d\theta}{dt} = -R_{\mu\nu}u^{\mu}u^{\nu} - \frac{1}{3}\theta^2 - 2\sigma^2 \tag{10}$$

$$\frac{d\theta}{dt} + \frac{1}{3}\theta^2 = -R_{\mu\nu}u^{\mu}u^{\nu} - 2\sigma^2$$

Let

$$\theta = \frac{1}{z} \cdot \frac{dz}{dt}$$

With

$$z = x^3, \theta = \frac{1}{x^3} \cdot 3x^2 \frac{dx}{dt} = \frac{3}{x}\frac{dx}{dt}$$

So,

$$\frac{d\theta}{dt} = -\frac{3}{x^2}\left(\frac{dx}{dt}\right)^2 + \frac{3}{x}\frac{d^2x}{dt^2}$$
$$\Rightarrow \frac{d\theta}{dt} + \frac{1}{3}\theta^2 = \frac{3}{x}\frac{d^2x}{dt^2}$$

Then (10) becomes

$$\frac{d^2x}{dt^2} + H(t)x = 0 \tag{11}$$

Where $H(t) = \frac{1}{3}\left(R_{\mu\nu}u^{\mu}u^{\nu} + 2\sigma^2\right)$

We utilize the Sturn comparison theorem to solve (11) comparing the distribution zeros of the solutions $u(t)$ and $v(t)$ of the equations.

$$\frac{d^2u}{dt^2} + G_1(t)u = 0 \tag{12}$$

$$\frac{d^2v}{dt^2} + G_2(t)v = 0 \tag{13}$$

where $G_1 \leq G_2$ in any interval (a, b). The theorem therefore states that if $u(t)$ has m zeros in $a < t < b$, $v(t)$ must have at least m zeros in that specific interval and k^{th} zero of $v(t)$ must be before $u(t)$'s k^{th} zero.

Now let,

$$C = minH(t) = min\frac{1}{3}\left(R_{\mu\nu}u^{\mu}u^{\nu} + 2\sigma^2\right)$$

and consider the following equation:

$$\frac{d^2x}{dt^2} + Cx = 0 \tag{14}$$

Now the general solution looks like this:

$$\mathrm{x} = \mathrm{C_1}\sin(\mathrm{C_2} + \sqrt{\mathrm{C}}\mathrm{t}) \tag{15}$$

A zero must have for x within this interval $0 \leq \sqrt{C}t \leq \frac{\pi}{2}$. So

$$0 \leq t \leq \frac{\pi}{2\sqrt{C}}$$

So, we can say that, $t_{max} = \frac{\pi}{2\sqrt{C}}$ From energy tensor and Einstein equation[14], we can write

$$R_{\mu\nu}u^{\mu}u^{\nu} = 8\pi G\left(T_{\mu\nu}u^{\mu}u^{\nu} + \frac{1}{2}T\right)$$
$$R_{\mu\nu}u^{\mu}u^{\nu} = 8\pi G(\rho + P)(u^4)^2 - 4\pi G\rho + 4\pi GP$$
$$R_{\mu\nu}u^{\mu}u^{\nu} \geq 4\pi G(\rho + 3P) \tag{16}$$

We postulate that the non-relativistic free gas of neutrinos, for which $P < \rho$, contributes the majority of the energy density in the current universe. So, we can write

$$R_{\mu\nu}u^{\mu}u^{\nu} \geq 4\pi G\rho$$
$$C = min\frac{1}{3}\left(R_{\mu\nu}u^{\mu}u^{\nu} + 2\sigma^2\right) \geq \frac{4}{3}\pi\rho G \tag{17}$$

Where, ρ is the present density of the universe. Hence the maximum possible age denoted by t_{max} is has to be

$$t_{max} = \frac{\pi}{2\sqrt{C}} = \frac{\pi}{2}\left(\frac{3}{4\pi\rho G}\right)^{\frac{1}{2}} = \pi\left(\frac{3}{16\pi\rho G}\right)^{\frac{1}{2}} \tag{18}$$

In radiation dominated models we can write $\mathrm{P} = \frac{\rho}{3}$.

Then equation (16) becomes

$$R_{\mu\nu}u^{\mu}u^{\nu} \geq 8\pi G\rho$$
$$t_{max} = \pi\sqrt{\frac{3}{32\pi\rho G}} \tag{19}$$

The observable galaxies show that the average mass density is around $10^{-30}gmcm^{-3}$.

If we consider matter's contribution, we must choose from a wide range of densities. The average matter density from all potential sources is estimated to range between 10^{-30} and $10^{-28}gmcm^{-3}$[6]. For $\rho = 10^{-30}gmcm^{-3}$

$$t_{max} = \pi\left(\frac{3}{16\pi\rho G}\right)^{\frac{1}{2}} = \pi\sqrt{\frac{3}{16\pi \times 10^{-30} \times 1000 \times 6.67 \times 10^{-11}}}\ sec = 9.43 \times 10^{10} years$$

For $\rho = 10^{-28} gmcm^{-3}$

$$t_{max} = \pi \left(\frac{3}{16\pi\rho G}\right)^{\frac{1}{2}} = \pi \sqrt{\frac{3}{16\pi \times 10^{-28} \times 1000 \times 6.67 \times 10^{-11}}}\ sec = .94 \times 10^{10} years$$

We notice that the upper bound on the universe's age fluctuates from around $.94 \times 10^{10}$ years to 9.4×10^{10} roughly years. The Table 1 shows the upper limit of the age changes with density.

Table 1. Maximum possible age of the universe as a function of mass-energy density

Matter density $10^{-30} gmcm^{-3}$	t_{max} 10^{10} years
1	9.4203
2	6.6612
3	5.4388
4	4.7102
5	4.2129
6	3.8458
7	3.5605
8	3.3306
9	3.1401
10	2.9790
20	2.1064
30	1.7199
40	1.4895
50	1.3322
60	1.2162
70	1.1259
80	1.0532
90	0.9930
100	0.9420

We can see this graphically in Figure 3.

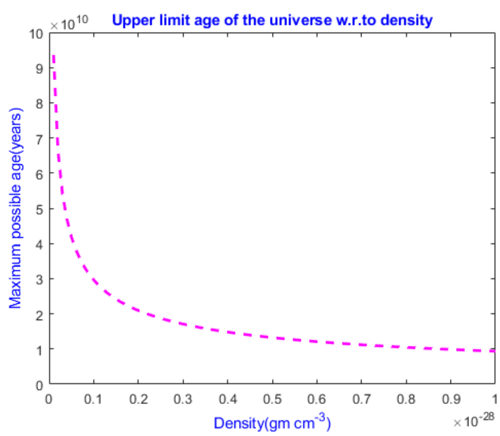

Fig. 3. Upper limit age of the Universe

In this figure, we observe that the maximum possible age depends on the density. The age is inversely proportional to density. Here density is measured as $gmcm^{-3}$ and age is measured as years. If density is high, then the possible age will be large and vice-versa.

6. Conclusion

Throughout this paper, it has been seen that our universe undergoes a flat expansion and also the progression of universe depends on the domination of radiation and matter developed by the Friedmann model. In the early universe when the expansion of was dominated by radiation, the density was proportional to R^{-4} and $R \propto t^{\frac{1}{2}}$ where R is the scale factor. Later on, when the expansion of our universe is dominated by matter, the density was proportional to R^{-3} and $R \propto t^{\frac{2}{3}}$. By Hubble's law, we have found the current age of our universe. We also have found the possible upper limit age of our universe with respect to density. The relation between age and density is inversely proportional to each other.

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Authors' Profiles

Oishi Khanam, M.S in Applied Mathematics, University of Dhaka, B.S in Applied Mathematics, University of Dhaka

Professor Dr. Md. Showkat Ali, Ph.D. in Mathematics, University of Glasgow, Scotland, UK; M.S in Mathematics, University of Dhaka, B.S in Mathematics, University of Dhaka; Professor and Chairman, Department of Applied Mathematics, University of Dhaka

Md Nirab Hossain, M.S in Applied Mathematics, University of Dhaka, B.S in Applied Mathematics, University of Dhaka

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