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# Bayesian Normal and T-K Approximations for Shape Parameter of Type-I Dagum Distribution

Hummara Sultan[a], Uzma Jan[b], S.P.Ahmad[*]

[a,b,*] *Department of Statistics, University of Kashmir, Srinagar, 190006, India*

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**Abstract**

Dagum distribution is a statistical distribution used closely for fitting income and wealth distributions. This distribution has wide application in fields like reliability theory survival analysis, actuarial sciences, and meteorological data. In this article, we obtained Bayes estimators for the shape parameter of Dagum distribution using approximation techniques like normal and T-K approximations. Moreover different informative priors have been considered and a simulation study and three real data sets have been considered to study the efficiency of obtained results.

**Index Terms:** Dagum distribution, Prior Distribution, Bayesian Statistics Normal approximation, T-K approximation.



## 1. Introduction

Camilo Dagum [6, 7] gave three-parameter type I and four-parameter type II and Type III distributions for fitting income and wealth distributions. However, the Dagum type I distribution has received more attention because the distribution has monotonically decreasing, upside-down bathtub, bathtub and then upside-down bathtub hazard rate for different values of parameters which led several authors to study the distribution in different fields Domma et al. [9, 11], Benjamin et al.[4]. Monroy et al.[16] used it for modeling tropospheric Ozone levels and Alwan et al. [2] worked with the Dagum distribution for assessing the reliability of an electrical system and for describing diameter in teak stands subjected to thinning at different ages. Different properties, characteristics and parameter estimation of Dagum distribution were studied by Kleiber and Kotz [15], Kleiber [14], Domma et al. [8, 10], Khan [13]. Broderick et al. [5] derived a new class of generalized Dagum distribution and studied its applications to income and life time data to illustrate the usefulness of the model. Aala Ahmed [1] proposed the estimates and asymptotic distribution of Dagum distribution. Tahir et al.

* Corresponding author.
E-mail address:

[23] defined a new life time model Weibull-Dagum distribution studied its structural properties and illustrated its potentiality by means of simulation study and real life applications.

The probability density function of Dagum distribution

$$f(y:\beta,\lambda,\gamma)=\beta\lambda\gamma\, y^{-\gamma-1}(1+\lambda\, y^{-\gamma})^{-\beta-1}\ \ ;\ \lambda,\beta,\gamma>0 \tag{1}$$

where $\gamma$ and $\beta$ are shape parameters and $\lambda$ is the scale parameter.

The likelihood function of (1.1) is given as

$$L(y:\beta,\lambda,\gamma)=\beta^n\,\lambda^n\,\gamma^n\exp(-(\gamma+1)\sum_{i=1}^{n}\ln y_i-(\beta+1)\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})) \tag{2}$$

The aim of our present study is to obtain the Bayes estimates of the shape parameter of Type-I Dagum distribution using normal approximation and T-K approximation techniques under different informative priors.

## 2. Bayesian Approximation Techniques of Posterior Modes

Bayesian inference provides a rational method for updating beliefs in light of new information. Bayesian analysis is based on the premise that all uncertainty should be modeled using probabilities and that statistical inference should be logical conclusions based on the laws of probability. It may be noted that posterior distribution takes a ratio that involves integration in the denominator and cannot be reduced to closed form. Hence the evaluation of the posterior expectation for obtaining the Bayes estimators will be tedious. Thus, we propose the use of Bayesian approximation techniques for obtaining Bayes estimates.

If the posterior distribution $\phi(\delta\,|\,x)$ is unimodal and roughly symmetric, it is convenient to approximate it by a normal distribution centered at the mode, yielding the approximation

$$\phi(\delta\,|\,x)\sim\hat{N}\left(\hat{\delta},\left[I(\hat{\delta})\right]^{-1}\right),\ \text{where } I(\hat{\delta})=-\frac{\partial^2\log P(\delta\,|\,y)}{\partial\delta^2} \tag{3}$$

If the mode, $\hat{\delta}$ is in the interior parameter space, then $I(\delta)$ is positive; if $\hat{\delta}$ is a vector parameter, then $I(\delta)$ is a matrix.

Tierney and Kadane [24] gave Laplace method to evaluate $E(h(\delta)\,|\,x)$ as

$$E(h(\delta)\,|\,x)\cong\frac{\phi^*}{\phi}\exp\{n\,h^*(\hat{\delta}^*)-n\,h(\hat{\delta})\},$$

where

$$-nh^*(\hat{\delta}^*)=\ln\pi(\delta\,|\,x)+\ln h(\delta),\ \hat{\phi}^2=-\left[n\,h''(\hat{\delta})\right]^{-1},\ \hat{\phi}^{*2}=-\left[nh''^*(\hat{\delta}^*)\right]^{-1}$$

Recently Sultan et al. [19, 20, 21, 22] obtained the Bayes estimates for Topp-Leone Distribution, Kumaraswamy distribution, generalized power function distribution, and generalized gamma distribution using Bayesian approximation techniques. Naqash et al. [17] proposed a Bayesian analysis of Dagum distribution for the complete sample under different loss function and priors.

## 3. Bayesian Normal Approximation for Shape Parameter of Type-I Dagum Distribution

In this section, the estimates of shape parameter under different priors are obtained using normal approximation technique.

The normal approximations for Type-I Dagum distribution under Mukherjee- Islam Prior $g(\beta) \propto \beta^{(b_1-1)}$, Gamma Prior $g(\beta) \propto \beta^{c_1-1} e^{-d_1\beta}; c_1, d_1 > 0$ and Inverse Levy Prior $g(\beta) \propto \beta^{-0.5} e^{-0.5a_2\beta}; a_2 > 0$ are obtained as:

Posterior density of $\beta$ under the Mukherjee- Islam Prior is

$$\pi(\beta \mid y) \propto \beta^{n+b_1-1} \exp(-(\gamma+1)\sum_{i=1}^{n} \ln y_i - (\beta+1)\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma})) \tag{4}$$

From which

$$\hat{\beta} = \frac{\partial \ln \pi(\beta \mid y)}{\partial \beta} = \frac{n+b_1-1}{\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma})}$$

and

$$\left[I(\hat{\beta})\right]^{-1} = \frac{(n+b_1-1)}{[\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma})]^2}$$

Therefore

$$\pi(\beta \mid y) \sim N\left(\frac{(n+b_1-1)}{[\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma})]}, \frac{(n+b_1-1)}{[\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma})]^2}\right)$$

The posterior density of $\beta$ under the gamma prior is

$$\pi(\beta \mid y) \propto \beta^{n+c_1-1} \exp(-(\beta+1)\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma}) - d_1\beta) \tag{5}$$

from which the posterior distribution can be approximated as

$$\pi(\beta \mid y) \sim N\left(\frac{(n+c_1-1)}{[d_1+\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma})]}, \frac{(n+c_1-1)}{[d_1+\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma})]^2}\right) \tag{6}$$

Similarly, under inverse levy prior the approximated posterior density of $\beta$ is as

$$\pi(\beta \mid y) \sim N\left(\frac{(n-1/2)}{[a_2/2+\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma})]}, \frac{(n-1/2)}{[a_2/2+\sum_{i=1}^{n} \ln(1+\lambda y_i^{-\gamma})]^2}\right) \tag{7}$$

## 4. T-K Approximation for shape parameter of Type-I Dagum Distribution

This section deals with calculating the Bayesian estimates of Dagum distribution using Laplace approximation technique introduced by Tierney and Kadane in 1986s.

Under Mukherjee- Islam Prior

$$nh(\beta) = \ln \pi(\beta \mid y) = (n + b_1 - 1)\ln\beta - (\beta + 1)\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})$$

From which $\hat{\beta} = \frac{\partial\, n\, h(\beta)}{\partial \beta} = \frac{n + b_1 - 1}{\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})}$ maximizes $nh(\beta)$. Since $\frac{\partial^2 n\, h(\beta)}{\partial \beta^2} = \frac{-(n + b_1 - 1)}{\beta^2} < 0$

Similarly, $nh^*(\beta) = \ln h(\beta) + \ln \pi(\beta \mid y) = (n + b_1)\ln\beta - (\beta + 1)\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})$

From which $\hat{\beta}^* = \frac{\partial\, n\, h^*(\beta)}{\partial \beta} = \frac{n + b_1}{\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})}$ maximizes $nh^*(\beta)$. Since $\frac{\partial^2 n\, h^*(\beta)}{\partial \beta^2} = \frac{-(n + b_1)}{\beta^2} < 0$

The maximum of $n\, h(\beta)$ and $n\, h^*(\beta)$ are given by

$$n\, h(\hat{\beta}) = (n + b_1 - 1)\ln\left(\frac{n + b_1 - 1}{\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})}\right) - (n + b_1 - 1)$$

$n\, h^*(\hat{\beta}^*) = (n + b_1)\ln\left(\frac{n + b_1}{\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})}\right) - (n + b_1)$ respectively. The estimates of variance are given by

$$\phi^{-2} = \left.\frac{-\partial^2 n\, h(\beta)}{\partial \beta^2}\right|_{\beta = \hat{\beta}} = \frac{[\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})]^2}{(n + b_1 - 1)} \Rightarrow \phi = \frac{\sqrt{(n + b_1 - 1)}}{[\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})]}$$

and

$$\phi^{*-2} = \left.\frac{-\partial^2 n\, h^*(\beta)}{\partial \beta^2}\right|_{\beta = \hat{\beta}^*} = \frac{[\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})]^2}{(n + b_1)} \Rightarrow \phi^* = \frac{\sqrt{(n + b_1)}}{[\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})]}$$

We have

$$E(\beta \mid y) \cong \frac{\phi^*}{\phi}\exp\{n\, h^*(\hat{\beta}^*) - n\, h(\hat{\beta})\} = \left(\frac{n + b_1}{n + b_1 - 1}\right)^{n + b_1 + 1/2} \frac{(n + b_1 - 1)}{\sum_{i=1}^{n} \ln(1 + \lambda\, y_i^{-\gamma})} e^{-1} \quad (8)$$

Similarly, we can approximate $E(\beta^2 \mid y)$ ; Here $h(\beta) = \beta^2$

$$nh^{**}(\beta) = (n+b_1+1)\ln\beta - (\beta+1)\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})$$

from which $\hat{\beta}^{**} = \frac{\partial\, n\, h^{**}(\beta)}{\partial\beta} = \frac{n+b_1+1}{\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})}$ and $\frac{\partial^2 n\, h^{**}(\beta)}{\partial\beta^2} = \frac{-(n+b_1+1)}{\beta^2} < 0$

Therefore, $\phi^{**} = \frac{\sqrt{(n+b_1+1)}}{[\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})]}$

Hence, second order moment is given as

$$E(\beta^2\mid y) \cong \frac{\phi^{**}}{\phi}\exp\{n\,h^{**}(\hat{\beta}^{**}) - n\,h(\hat{\beta})\} = \left(\frac{n+b_1+1}{n+b_1-1}\right)^{n+b_1+1/2} \frac{((n+b_1)^2-1)}{[\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})]^2}\, e^{-2}$$

$$\text{Thus, var} = \left(\frac{n+b_1+1}{n+b_1-1}\right)^{n+b_1+1/2} \frac{((n+b_1)^2-1)}{[\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})]^2}\, e^{-2} - \left[\left(\frac{n+b_1}{n+b_1-1}\right)^{n+b_1+1/2} \frac{(n+b_1-1)}{\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})}\, e^{-1}\right]^2 \quad (9)$$

Following the same procedure under gamma prior

We have

$$E(\beta\mid y) = \left(\frac{n+c_1}{n+c_1-1}\right)^{n+c_1+1/2} \frac{(n+c_1-1)}{d_1+\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})}\, e^{-1} \quad (10)$$

and

$$E(\beta^2\mid y) = \left(\frac{n+c_1+1}{n+c_1-1}\right)^{n+c_1+1/2} \frac{((n+c_1)^2-1)}{[d_1+\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})]^2}\, e^{-2}$$

$$\text{Thus, var} = \left(\frac{n+c_1+1}{n+c_1-1}\right)^{n+c_1+1/2} \frac{((n+c_1)^2-1)}{[d_1+\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})]^2}\, e^{-2} - \left[\left(\frac{n+c_1}{n+c_1-1}\right)^{n+c_1+1/2} \frac{(n+c_1-1)}{d_1+\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})}\, e^{-1}\right]^2 \quad (11)$$

Under inverse levy prior, we have

$$E(\beta\mid y) = \left(\frac{n+1/2}{n-1/2}\right)^{n} \frac{(n+1/2)}{a_2/2+\sum_{i=1}^{n}\ln(1+\lambda\, y_i^{-\gamma})}\, e^{-1} \quad (12)$$

and

$$E(\beta^2 \mid y) = \left(\frac{n+3/2}{n-1/2}\right)^n \frac{(n+3/2)}{a_2/2 + \sum_{i=1}^{n} \ln(1+\lambda\, y_i^{-\gamma})} e^{-2}$$

$$\text{Thus, var} = \left(\frac{n+3/2}{n-1/2}\right)^n \frac{(n+3/2)}{a_2/2 + \sum_{i=1}^{n} \ln(1+\lambda\, y_i^{-\gamma})} e^{-2} - \left[\left(\frac{n+1/2}{n-1/2}\right)^n \frac{(n+1/2)}{a_2/2 + \sum_{i=1}^{n} \ln(1+\lambda\, y_i^{-\gamma})} e^{-1}\right]^2 \qquad (13)$$

## 5. Simulation Study

For simulation study, three samples of sizes 25, 50 and 100 have been generated from Type I Dagum distribution to represent small, medium and large sizes using the R software to see the influence of various informative priors on the estimates of the shape parameter $\beta$ of type I Dagum distribution. In order to estimate the variability of the unknown shape parameter two approximation techniques namely Normal approximation and T-K approximation have been considered. The values of the hyper parameters have been randomly chosen as 0.5, 1.0 and 2.0. Similarly, the values of the known parameters have also been considered as 0.5, 1.0 and 2.0. The unknown shape parameter $\beta$ to be estimated has been fixed at 2.0. In order to observe the performance of shape parameter $\beta$ of Type I Dagum distribution, the experiment has been iterated 5000 times. The simulated results obtained have been presented in the tables 1 and 2 as given below with posterior variances enclosed in brackets.

Table 1. Posterior Estimates of Shape Parameter β of Type I Dagum Distribution Using Normal Approximation Technique for Simulated Data Sets

| $n$ | $\gamma$ | $\lambda$ | Mukherjee Islam prior | | | Gamma Prior | | | Inverse Levy Prior | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $b_1$=0.5 | $b_1$=1.0 | $b_1$=2.0 | $c_1$=$d_1$=0.5 | $c_1$=$d_1$=1.0 | $c_1$=$d_1$=2.0 | $a_2$=0.5 | $a_2$=1.0 | $a_2$=2.0 |
| 25 | 0.5 | 0.5 | 1.8298<br>(0.1366) | 1.8672<br>(0.1394) | 1.9419<br>(0.1450) | 1.7640<br>(0.1270) | 1.7374<br>(0.1207) | 1.6895<br>(**0.1097**) | 1.7963<br>(0.1317) | 1.7640<br>(0.1270) | 1.7027<br>(0.1183) |
| | 1 | 1 | 1.7743<br>(0.1285) | 1.8105<br>(0.1311) | 1.8829<br>(0.1363) | 1.7123<br>(0.1196) | 1.6883<br>(0.1140) | 1.6447<br>(**0.1040**) | 1.7428<br>(0.1239) | 1.7123<br>(0.1196) | 1.6545<br>(0.1117) |
| | 2 | 2 | 2.7159<br>(0.3010) | 2.7713<br>(0.3072) | 2.8822<br>(0.3195) | 2.5733<br>(0.2702) | 2.4948<br>(0.2489) | 2.3591<br>(**0.2140**) | 2.6427<br>(0.2850) | 2.5733<br>(0.2702) | 2.4449<br>(0.2439) |
| 50 | 0.5 | 0.5 | 1.0570<br>(0.0225) | 1.0677<br>(0.0228) | 1.0891<br>(0.0232) | 1.0459<br>(0.0220) | 1.0454<br>(0.0218) | 1.0444<br>(**0.0213**) | 1.0514<br>(0.0223) | 1.0459<br>(0.0220) | 1.0349<br>(0.0216) |
| | 1 | 1 | 2.2351<br>(0.1009) | 2.2577<br>(0.1019) | 2.3028<br>(0.1039) | 2.1857<br>(0.0965) | 2.1601<br>(0.0933) | 2.1121<br>(**0.0874**) | 2.2101<br>(0.0986) | 2.1857<br>(0.0965) | 2.1385<br>(0.0923) |
| | 2 | 2 | 3.8345<br>(0.2970) | 3.8733<br>(0.3000) | 3.9507<br>(0.3060) | 3.6915<br>(0.2753) | 3.5948<br>(0.2584) | 3.4207<br>(**0.2294**) | 3.7617<br>(0.2858) | 3.6915<br>(0.2753) | 3.5588<br>(0.2558) |
| 100 | 0.5 | 0.5 | 2.3898<br>(0.0574) | 2.4019<br>(0.0576) | 2.4259<br>(0.0582) | 2.3615<br>(0.0560) | 2.3455<br>(0.0550) | 2.3147<br>(**0.0530**) | 2.3756<br>(0.0567) | 2.3615<br>(0.0560) | 2.3338<br>(0.0547) |
| | 1 | 1 | 1.8706<br>(0.0351) | 1.8800<br>(0.0353) | 1.8988<br>(0.0356) | 1.8532<br>(0.0345) | 1.8453<br>(0.0340) | 1.8300<br>(**0.0331**) | 1.8618<br>(0.0348) | 1.8532<br>(0.0345) | 1.8361<br>(0.0338) |
| | 2 | 2 | 3.8988<br>(0.1527) | 3.9184<br>(0.1535) | 3.9575<br>(0.1550) | 3.8239<br>(0.1469 | 3.7706<br>(0.1421 | 3.6699<br>(**0.1333**) | 3.8609<br>(0.1498) | 3.8239<br>(0.1469) | 3.7518<br>(0.1414) |

Table 2. Posterior Estimates of Shape Parameter β of Type I Dagum Distribution Using T-K approximation Technique for Simulated Data Sets

| $n$ | $\gamma$ | $\lambda$ | Mukherjee Islam prior | | | Gamma Prior | | | Inverse Levy Prior | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $b_1$=0.5 | $b_1$=1.0 | $b_1$=2.0 | $c_1$=$d_1$=0.5 | $c_1$=$d_1$=1.0 | $c_1$=$d_1$=2.0 | $a_2$=0.5 | $a_2$=1.0 | $a_2$=2.0 |
| 25 | 0.5 | 0.5 | 2.2035 (0.1903) | 2.2467 (0.1940) | 2.3331 (0.2015) | 2.1122 (0.1749) | 2.0680 (0.1644) | 1.9893 (**0.1465**) | 2.1569 (0.1823) | 2.1122 (0.1749) | 2.0282 (0.1612) |
| | 1 | 1 | 2.2257 (0.1942) | 2.2693 (0.1980) | 2.3566 (0.2056) | 2.1326 (0.1783) | 2.0872 (0.1675) | 2.0064 (**0.1490**) | 2.1782 (0.1860) | 2.1326 (0.1783) | 2.0471 (0.1642) |
| | 2 | 2 | 5.9100 (1.3693) | 6.0259 (1.3962) | 6.2576 (1.4499) | 5.2963 (1.0997) | 4.8922 (0.9202) | 4.2758 (**0.6769**) | 5.5864 (1.2235) | 5.2963 (1.0997) | 4.7981 (0.9025) |
| 50 | 0.5 | 0.5 | 2.4878 (0.1225) | 2.5125 (0.1237) | 2.5617 (0.1261) | 2.4280 (0.1167) | 2.3945 (0.1124) | 2.3320 (**0.1045**) | 2.4576 (0.1195) | 2.4280 (0.1167) | 2.3710 (0.1113) |
| | 1 | 1 | 2.2955 (0.1043) | 2.3182 (0.1053) | 2.3636 (0.1074) | 2.2445 (0.0997) | 2.2174 (0.0964) | 2.1667 (**0.0902**) | 2.2697 (0.1020) | 2.2445 (0.0997) | 2.1957 (0.0954) |
| | 2 | 2 | 3.2921 (0.2145) | 3.3246 (0.2167) | 3.3898 (0.2209) | 3.1881 (0.2012) | 3.1212 (0.1910) | 2.9989 (**0.1729**) | 3.2393 (0.2077) | 3.1881 (0.2012) | 3.0906 (0.1891) |
| 100 | 0.5 | 0.5 | 3.1882 (0.1011) | 3.2041 (0.1016) | 3.2358 (0.1026) | 3.1385 (0.0980) | 3.1056 (0.0954) | 3.0428 (**0.0907**) | 3.1632 (0.0995) | 3.1385 (0.0980) | 3.0902 (0.0950) |
| | 1 | 1 | 1.8967 (0.0357) | 1.9062 (0.0359) | 1.9250 (0.0363) | 1.8790 (0.0351) | 1.8709 (0.0346) | 1.8550 (**0.0337**) | 1.8878 (0.0354) | 1.8790 (0.0351) | 1.8616 (0.0344) |
| | 2 | 2 | 3.5023 (0.1220) | 3.5197 (0.1226) | 3.5545 (0.1238) | 3.4423 (0.1179) | 3.4012 (0.1145) | 3.3229 (**0.1082**) | 3.4720 (0.1199) | 3.4423 (0.1179) | 3.3843 (0.1139) |

## 6. Applications

For justifying the results obtained in simulation study, three real data sets have been taken into consideration.

**Data Set I**: The first data set consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes Proschan [18]. The data is given as 194,413,90,74,55,23,97,50,359,50,130,487,57,102,15,14,10,57,320,261,51,44,9,254,493,33,18,209,41,58,60,48,56,87,11,102,12,5,14,14,29,37,186,29,104,7,4,72,270,283,7,61,100,61,502,220,120,141,22,603,35,98,54,100,11,181,65,49,12,239,14,18,39,3,12,5,32,9,438,43,134,184,20,386,182,71,80,188,230,152,5,36,79,59,33,246,1,79,3,27,201,84,27,156,21,16,88,130,14,118,44,15,42,106,46,230,26,59,153,104,20,206,5,66,34,29,26,35,5,82,31, 118,326,12,54,36,34,18,25,120,31,22,18,216,139,67,310,3,46,210,57,76,14,111,97,62,39,30,7,44,11,63,23,22,23,14,18,13,34,16,18,130,90,163,208,1,24,70,16,101,52,208,95,62,11,191,14,71.

**Data Set II:** The second data were first analyzed by Feigl and Zelen[12]. The data represent the survival times, in weeks, of 33 patients suffering from Acute Myelogenous Leukemia. The data are: 65,156, 100,134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43

**Data Set III:** The third real data set is a subset of the data reported by Bekker et al. [3], which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consisting of survival times (in years) for 46 patients are: 0.047,0.115, 0.121, 0.132, 0.164, 0.197,0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641,0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178,2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

Table 3. Posterior Estimates of Shape Parameter β of Type I Dagum Distribution using Normal Approximation Technique for three Real Data Sets

| | $\gamma$ | $\lambda$ | Mukherjee Islam prior | | | Gamma Prior | | | Inverse Levy Prior | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $b_1$=0.5 | $b_1$=1.0 | $b_1$=2.0 | $c_1$=$d_1$=0.5 | $c_1$=$d_1$=1.0 | $c_1$=$d_1$=2.0 | $a_2$=0.5 | $a_2$=1.0 | $a_2$=2.0 |
| ***Data Set I*** | 0.5 | 0.5 | 12.0145 (0.7698) | 12.0465 (0.7719) | 12.1106 (0.7760) | 11.6415 (0.7228) | 11.3211 (0.6817) | 10.7348 **(0.6097)** | 11.8250 (0.7457) | 11.6415 (0.7228) | 11.2910 (0.6799) |
| | 1 | 1 | 22.3486 (2.6637) | 22.4081 (2.6708) | 22.5273 (2.6850) | 21.0916 (2.3725) | 20.0217 (2.1322) | 18.1909 **(1.7508)** | 21.7019 (2.5118) | 21.0916 (2.3725) | 19.9685 (2.1266) |
| | 2 | 2 | 47.9877 (12.2817) | 48.1157 (12.3144) | 48.3716 (12.3799) | 42.5435 (9.6530) | 38.3107 (7.8069) | 31.9946 **(5.4161)** | 45.1019 (10.8490) | 42.5435 (9.6530) | 38.2088 (7.7862) |
| ***Data Set II*** | 0.5 | 0.5 | 6.8257 (1.4335) | 6.9307 (1.4556) | 7.1407 (1.4997) | 6.1770 (1.1740) | 5.7277 (0.9941) | 5.0285 (0.9941) | 6.4852 (1.2941) | 6.1770 (1.1740) | 5.6410 (**0.9791**) |
| | 1 | 1 | 7.2277 (1.6073) | 7.3389 (1.6321) | 7.5613 (1.6815) | 6.5044 (1.3017) | 6.0037 (1.0922) | 5.2335 (**0.8055**) | 6.8470 (1.4425) | 6.5044 (1.3017) | 5.9127 (1.0757) |
| | 2 | 2 | 8.4080 (2.1752) | 8.5374 (2.2087) | 8.7961 (2.2756) | 7.4450 (1.7054) | 6.7826 (1.3940) | 5.7967 (**0.9883**) | 7.8972 (1.9189) | 7.4450 (1.7054) | 6.6798 (1.3729) |
| ***Data Set III*** | 0.5 | 0.5 | 2.0588 (0.0952) | 2.0819 (0.0963) | 2.1282 (0.0984) | 2.0122 (0.0909) | 1.9898 (0.0879) | 1.9479 (**0.0824**) | 2.0352 (0.0930) | 2.0122 (0.0909) | 1.9677 (0.0870) |
| | 1 | 1 | 1.0394 (0.0242) | 1.0511 (0.0245) | 1.0744 (0.0250) | 1.0274 (0.0237) | 1.0271 (0.0234) | 1.0265 (**0.0229**) | 1.0334 (0.0239) | 1.0274 (0.0237) | 1.0157 (0.0231) |
| | 2 | 2 | 0.5482 (0.0067) | 0.5544 (0.0068) | 0.5667 (0.0069) | 0.5449 (0.0066) | 0.5476 (0.0066) | 0.5531 (0.0066) | 0.5465 (0.0067) | 0.5449 (0.0066) | 0.5415 (**0.0065**) |

Table 4. Posterior Estimates of Shape Parameter β of Type I Dagum Distribution using T-K approximation Technique for three Real Data Sets

| | $\gamma$ | $\lambda$ | Mukherjee Islam prior | | | Gamma Prior | | | Inverse Levy Prior | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $b_1$=0.5 | $b_1$=1.0 | $b_1$=2.0 | $c_1$=$d_1$=0.5 | $c_1$=$d_1$=1.0 | $c_1$=$d_1$=2.0 | $a_2$=0.5 | $a_2$=1.0 | $a_2$=2.0 |
| ***Data Set I*** | 0.5 | 0.5 | 12.0786 (0.7739) | 12.1106 (0.7760) | 12.1747 (0.7801) | 11.7036 (0.7266) | 11.3813 (0.6853) | 10.7917 **(0.6129)** | 11.8881 (0.7497) | 11.7036 (0.7266) | 11.3512 (0.6835) |
| | 1 | 1 | 22.4678 (2.6779) | 22.5274 (2.6850) | 22.6466 (2.6993) | 21.2041 (2.3852) | 20.1283 (2.1436) | 18.2872 **(1.7601)** | 21.8177 (2.5252) | 21.2041 (2.3852) | 20.0750 (2.1379) |
| | 2 | 2 | 48.2438 (12.3472) | 48.3717 (12.3799) | 48.6277 (12.4454) | 42.7705 (9.7045) | 38.5145 (7.8484) | 32.1639 **(5.4448)** | 45.3426 (10.9068) | 42.7705 (9.7045) | 38.4126 (7.8277) |
| ***Data Set II*** | 0.5 | 0.5 | 7.0363 (1.4776) | 7.1413 (1.4997) | 7.3513 (1.5438) | 6.3676 (1.2101) | 5.9018 (1.0242) | 5.1768 (**0.7655**) | 6.6853 (1.3339) | 6.3676 (1.2101) | 5.8150 (1.0092) |
| | 1 | 1 | 7.4507 (1.6568) | 7.5619 (1.6815) | 7.7842 (1.7310) | 6.7051 (1.3418) | 6.1861 (1.1253) | 5.3878 (**0.8292**) | 7.0582 (1.4869) | 6.7051 (1.3418) | 6.0951 (1.1088) |
| | 2 | 2 | 8.6674 (2.2421) | 8.7967 (2.2756) | 9.0554 (2.3425) | 7.6746 (1.7579) | 6.9887 (1.4363) | 5.9676 **(1.0173)** | 8.1408 (1.9780) | 7.6746 (1.7579) | 6.8859 (1.4152) |
| ***Data Set III*** | 0.5 | 0.5 | 2.1051 (0.0973) | 2.1283 (0.0984) | 2.1745 (0.1006) | 2.0575 (0.0930) | 2.0341 (0.0899) | 1.9903 (**0.0842**) | 2.0811 (0.0951) | 2.0575 (0.0930) | 2.0120 (0.0889) |
| | 1 | 1 | 1.0628 (0.0248) | 1.0745 (0.0250) | 1.0978 (0.0256) | 1.0505 (0.0242) | 1.0499 (0.0239) | 1.0488 (**0.0234**) | 1.0566 (0.0245) | 1.0505 (0.0242) | 1.0385 (0.0237) |
| | 2 | 2 | 0.5606 (0.0069) | 0.5667 (0.0069) | 0.5790 (0.0071) | 0.5571 (0.0068) | 0.5598 (0.0068) | 0.5651 (0.0067) | 0.5588 (0.0068) | 0.5571 (0.0068) | 0.5537 (**0.0066**) |

## 7. Conclusion

While comparing the estimates of the posterior variances of the shape parameter $\beta$ of Type I Dagum distribution using the three informative priors under the two approximation techniques, it is clearly evident that gamma prior is the best prior for the estimation of shape parameter especially when the value of the hyper parameters is taken as 2. This is because it has the minimum value of posterior variance in the simulation study which is apparent in the tables 1 and 2. Further, this prior has least value in the three real life data sets as well which confirms the efficiency of the Gamma prior as observed in the tables 3 and 4. It can also be noticed that the normal approximation technique can be preferred over the T-K approximation technique because of lesser posterior variance. Furthermore, the variability of the estimates in the tables 1 and 2 goes on decreasing as the sample size increases.

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**Authors' Profiles**

**S.P. Ahmad** is a Sr. Assistant Professor of Department of Statistics, at the University of Kashmir at Srinagar, Kashmir, India. He received his Ph.D degree in Statistics from University of Kashmir, India. His main research interests are in the areas of Bayesian Statistics and Probability distribution theory. He has published more than 45 publications.

**Hummara Sultan** is a Lecturer at Govt. Women's College Srinagar, Kashmir. She has completed her M.Sc. and Ph.D. from University of Kashmir. She has more than 12 papers published in refereed journals. Her main interests are Bayesian Statistics and Probability theory.

**Uzma Jan** is a research scholar in the Department of Statistics at the University of Kashmir. She has completed her P.G in the year 2013. She is currently pursuing her Ph.D. in the Department of Statistics. Her main interests are Bayesian Statistics and Probability distributions

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